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# Complexity of hereditarily decomposable continua

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**Abstract**

The purpose of this short note is to show that the set of hereditarily decomposable subcontinua of $I^n$ $(2 \leqslant n \leqslant \omega)$ is a coanalytic and non-Borel subset of $C(I^n)$, the space of all subcontinua of $I^n$ endowed with the Hausdorff metric. As a simple corollary to this result, we obtain that there is no model for $\mathcal{A}(I^n)$, the set of arcwise connected continua in $I^n$. © 2000 Elsevier Science B.V. All rights reserved.

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In recent years various examples of objects occurring naturally in analysis and topology have shown to be coanalytic and non-Borel [1–3,6,9]. The relevance of showing that a collection of objects is non-Borel is that it rules out characterizations of the collection which are "too simple" or "explicit enough" so that they can be expressed in terms of countable unions and intersections from closed or open sets. This is so because such characterizations would lead to a Borel definition of the collection.

In a conversation with the author, Tom Ingram stated that although there are "nice" characterizations for indecomposable continua [2], and even hereditarily indecomposable continua [12], there is not a satisfactory characterization for hereditarily decomposable continua. This led the author to investigate the descriptive complexity of the set of hereditarily decomposable continua. It will follow from the non-Borelness of the set of hereditarily decomposable continua that in some sense no "explicit" characterization of hereditarily decomposable continua exists. On the other hand, "explicit" characterizations of the collection of indecomposable continua and the collection of hereditarily indecomposable continua are justified from the fact that they form Borel sets (in particular $G_\delta$ sets).

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There is an extensive literature on the existence or nonexistence of universal continua or models for certain classes of continua (see [4]). As a simple corollary to our result, we give a simple descriptive set-theoretic proof of the fact that there is no model for $\mathcal{A}(I^n)$, the set of arcwise connected continua in $I^n$. The reader may refer to [1] for other results of this type.

We now review some terminology from descriptive set theory and continuum theory.

A set $M$ subset of a Polish space $X$ is *analytic* iff $M$ is the continuous image of a Borel subset of some Polish space. A set $M \subseteq X$ is *coanalytic* iff $X \backslash M$ is analytic. It is a classical result of Suslin that a set which is both analytic and coanalytic is Borel. A coanalytic set $M \subseteq X$ is *coanalytic complete* iff for every Polish space $Y$ and a coanalytic set $P \subseteq Y$, there exists a Borel function $f : Y \to X$ such that $f^{-1}(M) = P$. (We point out here that $f$ is not required to be onto.) As every uncountable Polish space contains sets which are coanalytic and non-Borel, coanalytic complete sets are non-Borel.

If $X$ is a metric space, then denote by $K(X)$ the space of all compact subsets of $X$ endowed with the Hausdorff metric, i.e., for $A, B \in K(X)$

$$d_H(A, B) = \min\left\{\varepsilon\colon\ A \subseteq \bigcup_{x \in B} B(x, \varepsilon) \text{ and } B \subseteq \bigcup_{x \in A} B(x, \varepsilon)\right\},$$

where $B(x, \varepsilon)$ denotes the ball in $X$ centered at $x$ with radius $\varepsilon$. Recall that if $X$ is a compact metric space, then $K(X)$ is a compact space [7].

Also recall that $\{0, 1\}^\omega$, the countable product of the discrete space $\{0, 1\}$, is homeomorphic to the Cantor set. We will use $\mu, \nu, \tau, \sigma$ to denote elements of $\{0, 1\}^{\leqslant\omega}$, i.e., finite or infinite sequence of 0's and 1's. We represent the length of a finite sequence $\nu$ by $|\nu|$. The $k$th term of $\nu$ is denoted $\nu(k)$, and if $\nu$ has length at least $n$ (or $\nu$ is infinite), then the truncated sequence $\{\nu(1), \nu(2), \ldots, \nu(n)\}$ is denoted by $\nu|_n$. If $\tau = \nu|_n$ for some $n$, then we say that $\nu$ is an extension of $\tau$. If $|\nu| = n$ and $i \in \{0, 1\}$, then we let $\nu i$ represent the sequence $\{\nu(1), \nu(2), \ldots, \nu(n), i\}$. Finally, let $\Sigma_0 = \{\emptyset\}$ and $\Sigma_n = \{\sigma \in \{0, 1\}^{<\omega}\colon |\sigma| = n\}$ for $n > 0$.

We will utilize the following classical result of Hurewicz in the proof of our main result.

**Lemma 1** (Hurewicz). *Let $D$ be a countable dense subset of $\{0, 1\}^\omega$. Then, $K(D)$ is a coanalytic complete subset of $K(\{0, 1\}^\omega)$.*

For proofs of this and other classical results of descriptive set theory stated above, the reader is referred to [5,7].

A *continuum* is a compact connected metric space. A continuum $M$ is *decomposable* means that $M$ is the union of two non-empty proper subcontinua. If $M$ is not decomposable, then $M$ is said to be *indecomposable*. A continuum $M$ is *hereditarily indecomposable* iff every subcontinuum of $M$ is indecomposable. A continuum $M$ is *hereditarily decomposable* iff every non-singleton subcontinuum of $M$ is decomposable. The reader may refer to [8] or [11] for examples of indecomposable and hereditarily indecomposable continua. We call continuum $M$ a *model* for collection $\mathcal{F}$ of continua if $M \in \mathcal{F}$ and every element of $\mathcal{F}$ is the continuous image of $M$.

Fix $2 \leqslant n \leqslant \omega$ and denote by $I^n$ the $n$-cube $[-1,1]^n$. We let $C(I^n)$ denote the set of all subcontinua of $I^n$. Note that $C(I^n)$ is a closed subset of $K(I^n)$ and hence is compact. Let $\mathcal{D}$ and $\mathcal{HD}$ denote the sets of decomposable and hereditarily decomposable subcontinua of $I^n$, respectively. Let $\mathcal{I}$ and $\mathcal{HI}$ denote the sets of non-singleton subcontinua of $I^n$ which are indecomposable and hereditarily indecomposable, respectively. That $\mathcal{HI}$ forms a dense $G_\delta$ subset of $C(I^n)$ is a well-known classical result of Mazurkiewicz [10].

A finite collection $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of open sets is a *chain* means that $G_i \cap G_j \neq \emptyset$ iff $|i-j| \leqslant 1$. The *mesh* of such a chain $\mathcal{G}$, denoted by *mesh* $\mathcal{G}$, is $\max\{\operatorname{diam}(G_i)\colon 1 \leqslant i \leqslant n\}$. Elements of chains are called *links*. Chain $\mathcal{G}'$ refines chain $\mathcal{G}$, denoted by $\mathcal{G}' \ll \mathcal{G}$, means that the closure of every link of $\mathcal{G}'$ is contained in some link of $\mathcal{G}$ and if $G \in \mathcal{G}$ there exists a $G' \in \mathcal{G}'$ such that

$$G' \cap \left[\bigcup \left(\mathcal{G} \setminus \{G\}\right)\right] = \emptyset.$$

Chain $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_{n'}\}$ *goes straight through* chain $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, denoted by $\mathcal{G}' \ll_s \mathcal{G}$, means that $\mathcal{G}' \ll \mathcal{G}$ and

- $G'_1 \subseteq G_1$, and
- if $G'_{i'} \subseteq G_i$ and $n' \geqslant j' \geqslant i'$, then $G'_{j'} \subseteq G_j$ for some $n \geqslant j \geqslant i$.

Chain $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_{n'}\}$ *follows* $z$*-pattern through* chain $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, denoted by $\mathcal{G}' \ll_z \mathcal{G}$, means that $\mathcal{G}' \ll \mathcal{G}$ and there are integers $1 < i' < j' < n'$ such that

- $\{G'_1, G'_2, \ldots, G'_{i'}\} \ll_s \mathcal{G}$,
- $\{G'_{j'}, G'_{j'-1}, \ldots, G'_{i'}\} \ll_s \mathcal{G}$, and
- $\{G'_{j'}, G'_{j'+1}, \ldots, G'_{n'}\} \ll_s \mathcal{G}$.

We now state lemmas necessary for our main result.

**Lemma 2.** *Suppose* $\{\mathcal{G}_n\}_{n=1}^{\infty}$ *is a sequence of chains such that*

- $\mathcal{G}_1$ *has at least two links,*
- $\lim_{n\to\infty} \operatorname{mesh}(\mathcal{G}_n) = 0$,
- *each link of* $\mathcal{G}_n$ *is connected, and*
- $\mathcal{G}_{n+1} \ll_s \mathcal{G}_n$ *for all* $n$.

*Then,* $\bigcap_{n=1}^{\infty}(\bigcup \mathcal{G}_n)$ *is an arc.*

This lemma is well known and its proof is essentially contained in the proof of Theorem 1 of [13, p. 84].

**Lemma 3.** *Suppose* $\{\mathcal{G}_n\}_{n=1}^{\infty}$ *is a sequence of chains such that*

- $\lim_{n\to\infty} \operatorname{mesh}(\mathcal{G}_n) = 0$,
- *each link of* $\mathcal{G}_n$ *is connected,*
- $\mathcal{G}_{n+1} \ll \mathcal{G}_n$ *for all* $n$*, and*
- *there is an increasing sequence of integers* $\{n_k\}_{k=1}^{\infty}$ *such that* $\mathcal{G}_{n_k+1} \ll_z \mathcal{G}_{n_k}$.

*Then,* $\bigcap_{n=1}^{\infty}(\bigcup \mathcal{G}_n)$ *is a non-singleton indecomposable continuum.*

This lemma is well known among continuum theorists as well. It also follows from the characterization of indecomposable continua given in [2].

Now we proceed to the main result of this paper.

**Lemma 4.** *$\mathcal{HD}$ is a coanalytic subset of $C(I^n)$.*

That $\mathcal{I}$ is a $G_\delta$ subset of $C(I^n)$ is well known and its proof may be found in [7] on p. 207. Now consider

$$\mathcal{A} = \{(L, M) \in C(I^n) \times C(I^n):\ L \subseteq M\}.$$

Then $\mathcal{A}$ is a closed subset of $C(I^n) \times C(I^n)$. Now let $\mathcal{B} = \mathcal{I} \times C(I^n)$. Then, $\mathcal{B}$ is a $G_\delta$ subset of $C(I^n) \times C(I^n)$. Hence, $\mathcal{A} \cap \mathcal{B}$ is a $G_\delta$ subset of $C(I^n) \times C(I^n)$ and $\pi_2(\mathcal{A} \cap \mathcal{B})$, the projection onto the second coordinate of $\mathcal{A} \cap \mathcal{B}$, is an analytic subset of $C(I^n)$. Observe that $\pi_2(\mathcal{A} \cap \mathcal{B})$ is precisely those continua in $C(I^n)$ which contain a non-singleton indecomposable continuum. As $\mathcal{S}$, the set of all singletons of $I^n$, is a closed subset of $C(I^n)$, we have that $\mathcal{HD} = C(I^n) \backslash (\pi_2(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{S})$ is coanalytic.

**Theorem 1.** *$\mathcal{HD}$ is a coanalytic complete subset of $C(I^n)$ and hence is non-Borel.*

We have already shown that $\mathcal{HD}$ is coanalytic. To show that $\mathcal{HD}$ is coanalytic complete, we will reduce $\mathcal{HD}$ to a known coanalytic complete set via a continuous map. More precisely, consider the dense set $D = \{\nu \in \{0,1\}^\omega:\ \nu \text{ is eventually } 0\}$. We will find a continuous map $f : K(\{0,1\}^\omega) \to C(I^2)$ such that $f^{-1}(\mathcal{HD}) = K(D)$. As $K(D)$ is a coanalytic complete subset of $K(\{0,1\}^\omega)$ (Lemma 1), it will follow that $\mathcal{HD}$ is a coanalytic complete subset of $C(I^n)$.

Before we construct our map $f$, we need to construct a collection of continua by induction. Since $I^2$ may be viewed as a subset of $I^n$, we will do our construction in $I^2$. Let $\mathcal{G}_\emptyset = \{G_1^\emptyset, G_2^\emptyset, \ldots, G_{l_\emptyset}^\emptyset\}$ be a chain such that each link of $\mathcal{G}_\emptyset$ is a circular disk in $I^2$, only the first link of $\mathcal{G}_\emptyset$ contains the origin and $l_\emptyset > 2$. Assume that a chain of circular disks $\mathcal{G}_\sigma = \{G_1^\sigma, G_2^\sigma, \ldots, G_{l_\sigma}^\sigma\}$ has been defined for all $\sigma \in \Sigma_n$ such that the following holds:

(1) $\operatorname{mesh}(\mathcal{G}_\sigma) < 2^{-n}$,
(2) only the first link of $\mathcal{G}_\sigma$ contains the origin,
(3) if $1 \leqslant i < n$, and $\sigma(i+1) = 0$, then $\mathcal{G}_{\sigma|_{i+1}} \ll_s \mathcal{G}_{\sigma|_i}$,
(4) if $1 \leqslant i < n$, and $\sigma(i+1) = 1$, then $\mathcal{G}_{\sigma|_{i+1}} \ll_z \mathcal{G}_{\sigma|_i}$, and
(5) if $\tau \in \Sigma_n$ and neither $\sigma$ nor $\tau$ is an extension of the other, then $G_i^\sigma \cap G_j^\tau = \emptyset$ unless $i = j = 1$.

Now we define $\mathcal{G}_\sigma$ for $\sigma \in \Sigma_{n+1}$. Fix $\nu \in \Sigma_n$. For $k \in \{0,1\}$, let $\mathcal{G}_{\nu k} = \{G_1^{\nu k}, G_2^{\nu k}, \ldots, G_{l_{\nu k}}^{\nu k}\}$ be a chain of circular disks which satisfies the following properties

- $\operatorname{mesh}(\mathcal{G}_{\nu k}) < 2^{-1(n+1)}$,
- only the first link of $\mathcal{G}_{\nu k}$ contains the origin,
- $\mathcal{G}_{\nu 0} \ll_s \mathcal{G}_\nu$, $\mathcal{G}_{\nu 1} \ll_z \mathcal{G}_\nu$, and
- $G_i^{\nu 0} \cap G_j^{\nu 1} = \emptyset$ unless $i = j = 1$.

Now we have defined $\mathcal{G}_{\nu 0}$ and $\mathcal{G}_{\nu 1}$. We may go through this process for all $\nu \in \Sigma_n$ and define $\mathcal{G}_{\nu 0}$ and $\mathcal{G}_{\nu 1}$ carefully enough so that condition (5) holds for all $\sigma, \tau \in \Sigma_{n+1}$. It is easy to check that conditions (1)–(4) are satisfied with respect to $n+1$ and $\Sigma_{n+1}$ as well.

Now for each $\sigma \in \{0,1\}^\omega$, let $M_\sigma = \bigcap_{i=0}^\infty (\bigcup \mathcal{G}_{\sigma|_i})$. Note that $M_\sigma$ is a continuum and from Lemmas 2 and 3 it follows that $M_\sigma$ is an arc if $\sigma \in D$, otherwise $M_\sigma$ is a non-singleton indecomposable continuum. Now we define $f : K(\{0,1\}^\omega) \to C(I^2)$. Let $A \in K(\{0,1\}^\omega)$. Define

$$f(A) = \bigcap_{i=1}^{\infty} \bigcup_{\sigma \in A} \Big(\bigcup \mathcal{G}_{\sigma|_i}\Big).$$

By the fashion in which $\mathcal{G}_\sigma$'s were defined, it follows that $f(A)$ is a continuum. Now we claim that $f(A) = \bigcup_{\sigma\in A} M_\sigma$. It is clear from the definition of $f(A)$ that $\bigcup_{\sigma\in A} M_\sigma \subseteq f(A)$. To show that $f(A) \subseteq \bigcup_{\sigma\in A} M_\sigma$, let $p \in f(A)$ be distinct from the origin. Let $n$ be a positive integer such that $2^{-n}$ is less than the distance from $p$ to the origin. From conditions (1), (2) and (5) we have that for each $m \geqslant n$ there is a unique $\mu$ such that $|\mu| = m$ and $p \in \bigcup \mathcal{G}_\mu$. Since $p \in f(A)$, we may choose $\{\tau_i\}_{i=1}^\infty$ in $\{0,1\}^\omega$ such that $\tau_i \in A$ and $p \in \bigcup \mathcal{G}_{\tau_i|_i}$. Note that $\tau_{i|j} = \tau_{j|j}$ for all $i \geqslant j \geqslant n$. Let $\tau$ be such that $\tau_{|i} = \tau_{i|i}$ for all $i \geqslant n$. As $A$ is closed, we have that $\tau \in A$. Now we have that $p \in M_\tau \subseteq \bigcup_{\sigma\in A} M_\sigma$, completing the proof of the claim.

Using the fact that $f(A) = \bigcup_{\sigma\in A} M_\sigma$, we observe that if $A \subseteq D$, then $f(A)$ is a continuum consisting of countably many arcs emanating from the origin and hence $f(A)$ is hereditarily decomposable. If $A$ contains $\sigma \in \{0,1\}^\omega \setminus D$ then $M_\sigma$ is a non-singleton indecomposable continuum and $f(A)$ is not hereditarily decomposable. Hence we have that $f^{-1}(\mathcal{HD}) = K(D)$.

Now the only thing that remains to be shown is that $f : K(\{0,1\}^\omega) \to C(I^2)$ is continuous. Consider the function $g : \{0,1\}^\omega \to C(I^2)$ defined by $g(\sigma) = M_\sigma$. Then $g$ is obviously continuous. As $\{0,1\}^\omega$ and $C(I^2)$ are compact, we have that the function $g' : K(\{0,1\}^\omega) \to K(C(I^2))$ defined by $g'(A) = \{g(\sigma)\colon \sigma \in A\}$ is continuous [8]. Also from [8], the union function $\bigcup : K(K(I^2)) \to K(I^2)$ defined by $\bigcup(B) = \bigcup\{C\colon C \in B\}$ is continuous. Since $f$ is the composition of $g'$ and $\bigcup$, we have that $f$ is continuous.

**Corollary 1.** *There is no model for $\mathcal{A}(I^n)$.*

We first note that the continuous image of an arcwise connected continuum is arcwise connected. To obtain a contradiction, assume that $M$ is a model for $\mathcal{A}(I^n)$. Then, $\mathcal{A}(I^n)$ is the set of all continuous images of $M$ into $I^n$. It is well known that the set of continuous images of a compact metric space into $I^n$ is an analytic subset of $K(I^n)$. Hence, $\mathcal{A}(I^n)$ is an analytic subset of $C(I^n)$. Now consider $f$ and $P = f(K(\{0,1\}^\omega))$ from the proof of Theorem 1. $P$ is a compact subset of $C(I^n)$ hence $P \cap \mathcal{A}(I^n)$ is analytic as well. However, $P \cap \mathcal{A}(I^n) = P \cap \mathcal{HD}$ because continua in $P$ are arcwise connected iff they are hereditarily decomposable. Therefore, $P \cap \mathcal{HD}$ is analytic as well as coanalytic and hence Borel, contradicting Theorem 1. Therefore, there is no model for $\mathcal{A}(I^n)$.

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